Example Question Answers

1. We have,
\[
\frac{4x^2 + 2}{x^3 - 4x^2 + 4x + 2x^2 - 8x + 8}
\]
Factorising the denominator gives,
\[
\frac{4x^2 + 2}{(x + 2)(x - 2)^2}
\]
We then separate this into partial fractions as follows,
\[
\frac{4x^2 + 2}{(x + 2)(x - 2)^2} = \frac{A}{x + 2} + \frac{B}{x - 2} + \frac{C}{(x - 2)^2}
\]
Equating the resulting numerators gives,
\[
4x^2 + 2 = A(x - 2)^2 + B(x + 2)(x - 2) + C(x + 2)
\]
We then substitute in \(x = -2\),
\[
16A = 18
\]
\[
A = \frac{18}{16} = \frac{9}{8}
\]
Similarly substituting in \(x = 2\) gives,
\[
4C = 18
\]
\[
C = \frac{18}{4} = \frac{9}{2}
\]
Now comparing coefficients of \(x^2\) we get,
\[
Ax^2 + Bx^2 = 4x^2
\]
Substituting in \(A = \frac{9}{2}\) gives,
\[
\frac{9}{2}x^2 + Bx^2 = 4x^2
\]
\[
B = \frac{23}{8}
\]
Therefore our final answer is,
\[
\frac{4x^2 + 2}{(x + 2)(x - 2)^2} = \frac{9}{8(x + 2)} + \frac{23}{8(x - 2)} + \frac{9}{2(x - 2)^2}
\]

2. We start from the equation in polar coordinates,
\[
4r^2 + r^3\cos \theta - 3 = r \sin \theta
\]
We know \(x^2 + y^2 = r^2\), \(x = r \cos \theta\) and \(y = r \sin \theta\), so we try to make our equation include terms of this form.
\[
4r^2 + r^2(r \cos \theta) - 3 = r \sin \theta
\]
So we can now substitute in for the terms we know.
\[
4(x^2 + y^2) + (x^2 + y^2)x - 3 = y
\]
\[
(x^2 + y^2)(4 + x) = y + 3
\]
And this is our final equation in terms of Cartesian coordinates \((x, y)\).
3. We have \( s = 103 - t^5 e^{-t} \), differentiating this gives,

\[
\frac{ds}{dt} = -5t^4 e^{-t} + t^5 e^{-t}
\]

\[
= e^{-t}t^4(t - 5)
\]

Setting this derivative to zero gives either \( t = 0 \) or \( t = 5 \) as our solutions.

\[
\frac{d^2 s}{dt^2} = -20t^3 e^{-t} + 5t^4 e^{-t} + 5t^4 e^{-t} - t^5 e^{-t}
\]

Substituting in \( t = 5 \) gives \( \frac{d^2 s}{dt^2} < 0 \) so the maximum occurs at \( t = 5 \).

We now substitute this back into our equation for \( s \) to get the maximum speed of the car,

\[
s = 103 - (5)^5 e^{-5}
\]

\[
= 103 - 21.056
\]

\[
= 81.9
\]

so the maximum speed is 81.9mph which occurs at time \( t = 5 \).

4. \[
\int_3^4 \int_1^2 x y^2 - 3x^2 + 4y \, dy \, dx = \int_3^4 \left[ \frac{xy^3}{3} - 3x^2y + 2y^2 \right]_1^2 \, dx
\]

\[
= \int_3^4 \frac{64x^4}{3} - 12x^3 + 35x^2 - x^3 - 2 \, dx
\]

\[
= \left[ \frac{64x^5}{15} - \frac{12x^4}{4} + \frac{35x^3}{3} - \frac{x^2}{6} - 2x \right]_1^4
\]

\[
= -16358
\]

5. (a) \( \nabla \cdot \mathbf{F} = \frac{d(y^2)}{dx} + \frac{d(-2x)}{dy} + \frac{d(xy)}{dz} = 0 \)

(b) \[
\nabla \times \mathbf{F} = \mathbf{i} \left( \frac{d(xy)}{dy} - \frac{d(-2x)}{dz} \right) - \mathbf{j} \left( \frac{d(xy)}{dx} - \frac{d(y^2)}{dz} \right) + \mathbf{k} \left( \frac{d(-2x)}{dx} - \frac{d(y^2)}{dy} \right)
\]

\[
= x\mathbf{\hat{i}} - y\mathbf{\hat{j}} - (2 + 2y)\mathbf{\hat{k}}
\]

(c) \[
\nabla f = \left( \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right) = (2xy, x^2, 3z^2)
\]

6. We have,

\[
\frac{d^2y}{dx^2} + 5 \frac{dy}{dx} - 14y = 10e^{3x}
\]

We first solve the homogeneous ODE by letting \( y = e^{mx} \), this gives:

\[
m^2 + 5m - 14 = 0
\]

\[
(m + 7)(m - 2) = 0
\]

Cont.
so \( m = -7 \) or \( m = 2 \) and our complementary function is,
\[
y_{CF} = Ae^{-7x} + Be^{2x}
\]
We now try \( y = Ce^{3x} \) to try and find the particular integral, this gives
\[
9Ce^{3x} + 15Ce^{3x} - 14Ce^{3x} = 10e^{3x}
\]
\[
10Ce^{3x} = 10e^{3x}
\]
\( C = 1 \)
so our particular integral is
\[
y_{PI} = e^{3x}
\]
Adding the CF and PI gives our general solution,
\[
y = Ae^{-7x} + Be^{2x} + e^{3x}
\]

7. \[
\begin{align*}
\frac{\partial f}{\partial x} &= 3x^2y - 4xz \\
\frac{\partial f}{\partial y} &= x^3 + z^2 \\
\frac{\partial f}{\partial z} &= 2yz - 2x^2 \\
\frac{\partial^2 f}{\partial x^2} &= 6xy - 4z \\
\frac{\partial^2 f}{\partial x \partial y} &= 3x^2 \\
\frac{\partial^2 f}{\partial y \partial z} &= 2z
\end{align*}
\]

8. We use De Moivre’s theorem!
\[
z = 2 \left[ \cos \left( \frac{7\pi}{7} \right) + j \sin \left( \frac{7\pi}{7} \right) \right]
\]
So
\[
z^7 = 2^7 \left[ \cos \left( \frac{7\pi}{7} \right) + j \sin \left( \frac{7\pi}{7} \right) \right] = 128 \left[ \cos \left( \pi \right) + j \sin \left( \pi \right) \right] = -128 + 128j
\]

9. A vector to the line is \( \left( \begin{array}{c} 2 \\ 5 \\ 1 \end{array} \right) \). A vector parallel to the line is \( \left( \begin{array}{c} 7 \\ 4 \\ 3 \end{array} \right) - \left( \begin{array}{c} 2 \\ 5 \\ 1 \end{array} \right) = \left( \begin{array}{c} 5 \\ -1 \\ 2 \end{array} \right) \).

So
\[
r = \left( \begin{array}{c} 2 \\ 5 \\ 1 \end{array} \right) + \lambda \left( \begin{array}{c} 5 \\ -1 \\ 2 \end{array} \right)
\]
is the parametric form of the equation.

This gives us the system of equations: \( x = 2 + 5\lambda \), \( y = 5 - \lambda \), \( z = 1 + 2\lambda \)

Rearrange these to make \( \lambda \) the subject to obtain:
\[
\lambda = \frac{x - 2}{5}
\]
Hence
\[
\frac{x - 2}{5} = 5 - y = \frac{z - 1}{2}
\]
is the Cartesian vector equation of the line.
10. (a) 

\[ 3A - B = 3 \begin{pmatrix} 2 & 1 \\ -6 & 7 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -5 & 9 \end{pmatrix} = \begin{pmatrix} 6 - 1 & 3 - 0 \\ -18 + 5 & 21 - 9 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ -13 & 12 \end{pmatrix} \]

(b) 

\[ AB = \begin{pmatrix} 2 & 1 \\ -6 & 7 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -5 & 9 \end{pmatrix} = \begin{pmatrix} (2 \times 1) + (1 \times -5) & (2 \times 0) + (1 \times 9) \\ (-6 \times 1) + (7 \times -5) & (-6 \times 0) + (7 \times 9) \end{pmatrix} = \begin{pmatrix} -3 & 9 \\ -41 & 63 \end{pmatrix} \]

(c) 

\[ B^{-1} = \frac{1}{\det(B)} \begin{pmatrix} 9 & 0 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{9 - 0} & 0 \\ \frac{1}{9 - 0} & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & 0 \\ \frac{1}{5} & \frac{1}{5} \end{pmatrix} \]

(d) We start by finding the eigenvalues of \( A \),

\[ |A - \lambda I| = \begin{vmatrix} 2 - \lambda & 1 \\ -6 & 7 - \lambda \end{vmatrix} = (2 - \lambda)(7 - \lambda) - (1)(-6) = \lambda^2 - 9\lambda + 20 = (\lambda - 4)(\lambda - 5) = 0 \]

This gives \( \lambda = 4 \) or \( \lambda = 5 \), which are therefore the eigenvalues of \( A \).

Next we find the eigenvalues of \( B \),

\[ |B - \lambda I| = \begin{vmatrix} 1 - \lambda & 0 \\ -5 & 9 - \lambda \end{vmatrix} = (1 - \lambda)(9 - \lambda) - (0)(-5) = \lambda^2 - 10\lambda + 9 = (\lambda - 9)(\lambda - 1) = 0 \]

This gives \( \lambda = 9 \) or \( \lambda = 1 \), which are therefore the eigenvalues of \( B \).
11. (a) We use the IVT to show that a root exists between $x = -2$ and $x = -3$,

$$
\begin{align*}
    f(-2) &= (-2)^3 - 5(-2) + 2 \\
    &= -8 + 10 + 2 \\
    &= 4 \\

    f(-3) &= (-3)^3 - 5(-3) + 2 \\
    &= -27 + 15 + 2 \\
    &= -10
\end{align*}
$$

so $f(-2)f(-3) = 4 \times -10 = -40 < 0$ therefore there is a root between $-2$ and $-3$.

(b) The formula for Newton Raphson is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

so substituting in our $f$ we get

$$
\begin{align*}
    x_2 &= -2 - \frac{(-2)^3 - 5(-2) + 2}{3(-2)^2 - 5} \\
    &= -2 - \frac{4}{7} \\
    &= -\frac{18}{7} \\
    &= -2.571
\end{align*}
$$

Repeating this process we get $x_3 = -2.426$.

12. (a) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.5 - 0.15 = 0.75$

(b) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) = 0.4 + 0.5 + 0.5 - 0.15 - 0.2 - 0.2 + 0.1 = 0.95$

(c) Recall that $P(A \cap C') + P(A \cap C) = P(A)$. So $P(A \cap C') = P(A) - P(A \cap C)$

Then $P(A) + P(C') - P(A \cap C') = 0.4 + (1 - 0.5) - (0.4 - 0.2) = 0.7$

(d) $P(A \cup B | C) = \frac{P((A \cup B) \cap P(C))}{P(C)} = \frac{P((A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{0.5}$

$$
\begin{align*}
    &= \frac{0.2 + 0.2 - 0.1}{0.5} = \frac{0.3}{0.5} = 0.6
\end{align*}
$$

13. (a) A type I error is when you reject the null hypothesis when it is true whereas a type II error is a failure to reject the null hypothesis when it is false.

(b) Let $\theta$ denote the expected weight. We test the null hypothesis $H_0 : \theta = 50$ against the alternative hypothesis $H_1 : \theta \neq 50$ using a z-test. The test statistic is

$$
    z = \frac{\bar{x} - 50}{\sigma} = \frac{49.95 - 50}{0.1} = -2.0
$$

with null distribution $N(0, 1)$. We only reject $H_0$ in favour of $H_1$ if $|z| > c$ and here $c = 1.96$, the upper 2.5% quantile of the $N(0, 1)$ distribution. Since $|z| > c$, we reject $H_0$ and conclude that there is sufficient evidence to disprove the companys claim.

The End.