Example Question Answers

1. We have,

$$\frac{4x^2+2}{x^3-4x^2+4x+2x^2-8x+8}$$

Factorising the denominator gives,

$$\frac{4x^2+2}{(x+2)(x-2)^2}$$

We then separate this into partial fractions as follows,

$$\begin{aligned}\frac{4x^2+2}{(x+2)(x-2)^2} &= \frac{A}{x+2}+\frac{B}{x-2}+\frac{C}{(x-2)^2}\\ &= \frac{A(x-2)^2+B(x+2)(x-2)+C(x+2)}{(x+2)(x-2)^2}\end{aligned}$$

Equating the resulting numerators gives,

$$4x^2+2 = A(x-2)^2+B(x+2)(x-2)+C(x+2)$$

We then substitute in $x=-2$,

$$\begin{aligned}16A &= 18\\ A &= \frac{18}{16}=\frac{9}{8}\end{aligned}$$

Similarly substituting in $x=2$ gives,

$$\begin{aligned}4C &= 18\\ C &= \frac{18}{4}=\frac{9}{2}\end{aligned}$$

Now comparing coefficients of x^2 we get,

$$Ax^2+Bx^2=4x^2$$

Substituting in $A=\frac{9}{2}$ gives,

$$\begin{aligned}\frac{9}{2}x^2+Bx^2 &= 4x^2\\ B &= \frac{23}{8}\end{aligned}$$

Therefore our final answer is,

$$\frac{4x^2+2}{(x+2)(x-2)^2}=\frac{9}{8(x+2)}+\frac{23}{8(x-2)}+\frac{9}{2(x-2)^2}$$

2. We start from the equation in polar coordinates,

$$4r^2+r^3\cos\theta-3=r\sin\theta$$

We know $x^2+y^2=r^2$, $x=r\cos\theta$ and $y=r\sin\theta$, so we try to make our equation include terms of this form.

$$4r^2+r^2(r\cos\theta)-3=r\sin\theta$$

So we can now substitute in for the terms we know.

$$\begin{aligned}4(x^2+y^2)+(x^2+y^2)x-3 &= y\\ (x^2+y^2)(4+x) &= y+3\end{aligned}$$

And this is our final equation in terms of Cartesian coordinates (x,y).

3. We have $s = 103 - t^5 e^{-t}$, differentiating this gives,

$$\frac{ds}{dt} = -5t^4 e^{-t} + t^5 e^{-t}$$
$$= e^{-t} t^4 (t-5)$$

Setting this derivative to zero gives either $t = 0$ or $t = 5$ as our solutions.

$$\frac{d^2 s}{dt^2} = -20t^3 e^{-t} + 5t^4 e^{-t} + 5t^4 e^{-t} - t^5 e^{-t}$$
$$= -20t^3 e^{-t} + 10t^4 e^{-t} - t^5 e^{-t}$$

Substituting in $t = 5$ gives $\frac{d^2 s}{dt^2} < 0$ so the maximum occurs at $t = 5$.

We now substitute this back into our equation for s to get the maximum speed of the car,

$$s = 103 - (5)^5 e^{-5}$$
$$= 103 - 21.056$$
$$= 81.9$$

so the maximum speed is 81.9mph which occurs at time $t = 5$.

4.

$$\int_{x=1}^{3} \int_{y=1}^{4x} xy^2 - 3x^2 + 4y \; dydx = \int_{x=1}^{3} \left[\frac{xy^3}{3} - 3x^2 y + 2y^2 \right]_1^{4x} \; dx$$
$$= \int_{x=1}^{3} \frac{64x^4}{3} - 12x^3 + 35x^2 - \frac{x}{3} - 2 \; dx$$
$$= \left[\frac{64x^5}{15} - \frac{12x^4}{4} + \frac{35x^3}{3} - \frac{x^2}{6} - 2x \right]_1^3$$
$$= \frac{16358}{15}$$

5. (a) $\bigtriangledown \cdot \mathbf{F} = \frac{d(y^2)}{dx} + \frac{d(-2x)}{dy} + \frac{d(xy)}{dz} = 0$

(b)

$$\bigtriangledown \times \mathbf{F} = \hat{\mathbf{i}} \left(\frac{d(xy)}{dy} - \frac{d(-2x)}{dz} \right) - \hat{\mathbf{j}} \left(\frac{d(xy)}{dx} - \frac{d(y^2)}{dz} \right) + \hat{\mathbf{k}} \left(\frac{d(-2x)}{dx} - \frac{d(y^2)}{dy} \right)$$
$$= x\hat{\mathbf{i}} - y\hat{\mathbf{j}} - (2 + 2y)\hat{\mathbf{k}}$$

(c)

$$\bigtriangledown f = \left(\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right) = (2xy, x^2, 3z^2)$$

6. We have,

$$\frac{d^2 y}{dx^2} + 5\frac{dy}{dx} - 14y = 10e^{3x}$$

We first solve the homogeneous ODE by letting $y = e^{mx}$, this gives:

$$m^2 + 5m - 14 = 0$$
$$(m + 7)(m - 2) = 0$$

Cont.

so $m = -7$ or $m = 2$ and our complementary function is,

$$y_{CF} = Ae^{-7x} + Be^{2x}$$

We now try $y = Ce^{3x}$ to try and find the particular integral, this gives

$$\begin{aligned} 9Ce^{3x} + 15Ce^{3x} - 14Ce^{3x} &= 10e^{3x} \\ 10Ce^{3x} &= 10e^{3x} \\ C &= 1 \end{aligned}$$

so our particular integral is

$$y_{PI} = e^{3x}$$

Adding the CF and PI gives our general solution,

$$y = Ae^{-7x} + Be^{2x} + e^{3x}$$

7. $\dfrac{\partial f}{\partial x} = 3x^2y - 4xz$
   $\dfrac{\partial f}{\partial y} = x^3 + z^2$
   $\dfrac{\partial f}{\partial z} = 2zy - 2x^2$
   $\dfrac{\partial^2 f}{\partial x^2} = 6xy - 4z$
   $\dfrac{\partial^2 f}{\partial x \partial y} = 3x^2$
   $\dfrac{\partial^2 f}{\partial y \partial z} = 2z$

8. We use De Moivre's theorem!

$$z = 2\left[\cos\left(\frac{\pi}{7}\right) + j\sin\left(\frac{\pi}{7}\right)\right]$$

So

$$z^7 = 2^7\left[\cos\left(\frac{7\pi}{7}\right) + j\sin\left(\frac{7\pi}{7}\right)\right] = 128\left[\cos\left(\pi\right) + j\sin\left(\pi\right)\right] = -128 + 128j$$

9. A vector to the line is $\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$. A vector parallel to the line is $\begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$.

   So $\mathbf{r} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$ is the parametric form of the equation.

   This gives us the system of equations: $x = 2 + 5\lambda$, $y = 5 - \lambda$, $z = 1 + 2\lambda$

   Rearrange these to make λ the subject to obtain: $\lambda = \dfrac{x-2}{5} = 5 - y = \dfrac{z-1}{2}$
   Hence $\dfrac{x-2}{5} = 5 - y = \dfrac{z-1}{2}$ is the Cartesian vector equation of the line.

Cont.

10. (a)

$$
\begin{aligned}
3A - B &= 3\begin{pmatrix} 2 & 1 \\ -6 & 7 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -5 & 9 \end{pmatrix} \\
&= \begin{pmatrix} 6-1 & 3-0 \\ -18+5 & 21-9 \end{pmatrix} \\
&= \begin{pmatrix} 5 & 3 \\ -13 & 12 \end{pmatrix}
\end{aligned}
$$

(b)

$$
\begin{aligned}
AB &= \begin{pmatrix} 2 & 1 \\ -6 & 7 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -5 & 9 \end{pmatrix} \\
&= \begin{pmatrix} (2\times 1) + (1 \times -5) & (2 \times 0) + (1 \times 9) \\ (-6 \times 1) + (7 \times -5) & (-6 \times 0) + (7 \times 9) \end{pmatrix} \\
&= \begin{pmatrix} -3 & 9 \\ -41 & 63 \end{pmatrix}
\end{aligned}
$$

(c)

$$
\begin{aligned}
B^{-1} &= \frac{1}{det(B)} \begin{pmatrix} 9 & 0 \\ 5 & 1 \end{pmatrix} \\
&= \frac{1}{9-0} \begin{pmatrix} 9 & 0 \\ 5 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ \frac{5}{9} & \frac{1}{9} \end{pmatrix}
\end{aligned}
$$

(d) We start by finding the eigenvalues of A,

$$
\begin{aligned}
|A - \lambda I| &= \begin{vmatrix} 2-\lambda & 1 \\ -6 & 7-\lambda \end{vmatrix} \\
&= (2-\lambda)(7-\lambda) - (1)(-6) \\
&= \lambda^2 - 9\lambda + 20 \\
&= (\lambda - 4)(\lambda - 5) \\
&= 0
\end{aligned}
$$

This gives $\lambda = 4$ or $\lambda = 5$, which are therefore the eigenvalues of A.

Next we find the eigenvalues of B,

$$
\begin{aligned}
|B - \lambda I| &= \begin{vmatrix} 1-\lambda & 0 \\ -5 & 9-\lambda \end{vmatrix} \\
&= (1-\lambda)(9-\lambda) - (0)(-5) \\
&= \lambda^2 - 10\lambda + 9 \\
&= (\lambda - 9)(\lambda - 1) \\
&= 0
\end{aligned}
$$

This gives $\lambda = 9$ or $\lambda = 1$, which are therefore the eigenvalues of B.

Cont.

11. (a) We use the IVT to show that a root exists between $x = -2$ and $x = -3$,

$$\begin{aligned} f(-2) &= (-2)^3 - 5(-2) + 2 \\ &= -8 + 10 + 2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} f(-3) &= (-3)^3 - 5(-3) + 2 \\ &= -27 + 15 + 2 \\ &= -10 \end{aligned}$$

so $f(-2)f(-3) = 4 \times -10 = -40 < 0$ therefore there is a root between -2 and -3.

(b) The formula for Newton Raphson is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

so substituting in our f we get

$$\begin{aligned} x_2 &= -2 - \frac{(-2)^3 - 5(-2) + 2}{3(-2)^2 - 5} \\ &= -2 - \frac{4}{7} \\ &= -\frac{18}{7} \\ &= -2.571 \end{aligned}$$

Repeating this process we get $x_3 = -2.426$.

12. (a) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.5 - 0.15 = 0.75$

(b) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap c) - P(B \cap C) + -P(A \cap B \cap C) = 0.4 + 0.5 + 0.5 - 0.15 - 0.2 - 0.2 + 0.1 = 0.95$

(c) Recall that $P(A \cap C') + P(A \cap C) = P(A)$. So $P(A \cap C') = P(A) - P(A \cap C)$
Then $P(A) + P(C') - P(A \cap C') = 0.4 + (1 - 0.5) - (0.4 - 0.2) = 0.7$

(d) $P(A \cup B|C) = \dfrac{P((A \cup B) \cap P(C))}{P(C)} = \dfrac{P((A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{0.5}$
$= \dfrac{P((A \cup B) \cap P(C))}{0.5} = \dfrac{0.2 + 0.2 - 0.1}{0.5} = \dfrac{0.3}{0.5} = 0.6$

13. (a) A type I error is when you reject the null hypothesis when it is true whereas a type II error is a failure to reject the null hypothesis when it is false.

(b) Let θ denote the expected weight. We test the null hypothesis $H_0 : \theta = 50$ against the alternative hypothesis $H_1 : \theta \neq 50$ using a z-test. The test statistic is

$$z = \frac{\bar{x} - 50}{\frac{\sigma}{\sqrt{n}}} = \frac{49.95 - 50}{\frac{0.1}{\sqrt{16}}} = -2.0$$

with null distribution $N(0,1)$. We only reject H_0 in favour of H_1 if $|z| > c$ and here $c = 1.96$, the upper 2.5% quantile of the $N(0,1)$ distribution. Since $|z| > c$, we reject H_0 and conclude that there is sufficient evidence to disprove the companys claim.

The End.