PHILOSOPHICAL PERSPECTIVES ON PROOF IN MATHEMATICS EDUCATION

LEE, Joong Kwoen

Dept. of Math. Ed., Dongguk University, 26 Pil-dong, Jung-gu, Seoul 100-715, Korea; joonglee@dgu.edu

ABSTRACT

This research reviewed literature on proof in mathematics education. Several views of proof classified and identified such as psychological approach, (Platonism, empiricism), structural approach, (logicism, formalism, intuitionism), social approach, (ontology, axiomatic systems). All these views of proof are valuable in mathematics education society. The concept of proof can be found in the form of analytic knowledge not of constructive knowledge. Human beings developed their knowledge in the sequence of constructive knowledge to analytic knowledge. Therefore, in mathematics education, the curriculum of mathematics should involve the process of cognitive knowledge development.

INTRODUCTION

Proof has been considered a fundamental part of mathematical practice science from ancient times. It was introduced into the mathematics curriculum long ago and has been given an important status in teaching Euclidean geometry. During the new math movement from the 1950s to 1960s, proof was stressed in other areas of mathematics as well. However students have serious difficulties in learning proof (Williams, 1979; Senk 1985). There have been lots of attempts to improve students' learning of proof (Bittinger, 1968; Brennan, 1976; Carlson, 1971; Summa, 1981)

Conceptions of mathematics have recently been changing in mathematics education society. Lakatos (1976) and Tymoczko (1985) pointed out the crucial importance of informal and social aspects of mathematical inquiry in the history and philosophy of mathematics. This new perspective of mathematics has been integrated into the recent constructivist trend in mathematics education (Bauersfeld, 1988; Confrey, 1987; Steffe & Cobb, 1983; von Glasersfeld, 1987; Weinberg & Gavelek, 1987).

PROOF IN MATHEMATICS EDUCATION

Mathematical proof has come to be viewed as social practice and analyzed for its intra - and interpersonal meanings in mathematics education (Sekiguchi, 1991). Research on the learning of proof from this new perspective toward mathematics is at the beginning stage, and very few studies have been done. Since the evolution of early practical and empirical mathematical knowledge into a systematic deductive mathematics in ancient Greece, proof has been given a central status in modern mathematics.

Conceptions of proof have been embedded in a problem of general epistemology (Lakatos, 1978). Until the recently, epistemology had been argued in a psychological framework, where the process of one’s arriving at a state of a true belief was the focus. In this framework, proof was conceived of as a certain legitimate process for an individual to reach a state of true belief (Cussins, 1987).

The concept of proof was formed exclusively in the tradition of Platonism. By the means of Platonism, mathematical objects are abstract entities in the Platonic universe. They exist independently of any human activity, and each of them has an unchangeable nature, as an essence (Popper, 1950).

The proof procedure in Platonism consists in the derivation of a theorem by following valid inferences from trivial truths (axioms, or postulates), perfectly clear descriptions of essences (definitions), and previously derived theorems. This exclusive reliance on the intuition in establishing valid proofs became common among mathematics education philosophers.

Pascal conceived of proof as a psychological process. He said that the art of persuading is simply the process of perfect methodical proofs, consists of three essential parts: of defining the terms of which we should avail ourselves by clear definitions; of proposing principles or evident axioms to prove the thing in question; and of always mentally substituting in the demonstrations the definition in the place of the thing defined. In his “art of persuasion”, we can see that he relied his conception on Euclidean method. Though he mentioned that persuading others may also require the art of “pleasing”, he explained only the art of “convincing”. His art lacked any concern for the other persons whom one whishes to persuade and concentrated on how to convince oneself.

Descartes’ method was also formulated psychologically and reflected Platonic views. Proof consisted in deducing from principles known to be true by a continuous and uninterrupted movement of thought, with clear intuition of each point. He specified the mental capacities needed to complete a proof: intuition, together with memory, which reminds one that each step of deduction is linked to the next.

Nagel (1950) had a psychological view of logic. He claimed that the necessity of a theorem does not come from a priori or logical impossibility of its negation. Using associated psychology, he mentioned that the necessity comes from the psychological difficulty in conceiving anything as possible, which is in contradiction to long, established and familiar experience, or even to old familiar habits of thought.

Nagel (1950) rejected Platonism and suggested empiricism in mathematics. ‘Two and one is equal to three’ or ‘two straight lines cannot make a space’ means statements about objects not in a Platonic universe but in the real world. These arguments are inductive generalizations from perceptual experiences or direct observations.

He noted that a real inference does not processed in syllogistic form, but from particulars to particulars in the medium of induction. He demonstrated the psychological process of finding an illustration of a geometrical theorem to show how inductive inferences are involved in the process. Therefore, he claimed that mathematical proof is not a deductive process but an inductive one. He tried to base the certainty of mathematics on that of direct observation and induction.

He took a nominal position for definition. He refused the Platonic concept of definition and accepted as a genuine concept of definition of a name, which explains the use or meaning of the name. The definition is a mere identical proposition which gives information only about the use of language, and from which no conclusions affecting matters of fact can possibly be drawn (Nagel, 1950).

Around the end of the last century, Frege (1950) criticized preceding psychological view of mathematical proof. He argued that a psychological approach is not appropriate for establishing mathematics as an objective science. In his investigation of the "foundations of arithmetic", he adopted the methodological principle, which always separate sharply the psychological view from the logical view, and it also segregate the subjective view from the objective view. He insisted that mathematics does not deal with ideas – mental pictures or mental construction – but concepts, which are objective entities, neither psychological nor physical objects.

He described the aim of proof that is, in fact, not merely to place the truth of a proposition beyond all doubt, bout also to afford us insight into the dependence of truths upon one another. The former refers to finding a convincing proposition, and the latter refers to justifying it logically. It is not uncommonly happens that we first discover the content of a proposition, and only later give the rigorous proof of it; and often this same proof also reveals more precisely the conditions restricting the validity of the original proposition. In general, therefore, we should separate the problem of how arrive at the content of a judge from the problem of how its assertion is to be justified (Frege, 1950).

Proof came to be conceived of not as a psychological process but as a system of propositions being logically interconnected. Frege's conception of proof can be called the structural conception of proof. He concentrated his efforts on where proof was considered as a justification device rather than as a medium to enable an individual to reach a true belief. Frege's conception of proof can be called the structural conception of proof. It has been further developed in the logicism and formalism of the foundations of mathematics.

The main idea of the logic program is that all mathematical concepts are defined by logical terms and all mathematical propositions are deduced from logical truths, so that all mathematics reduced to logic. Russell(1937) comments that all pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts, and all its properties are deducible from a very small number of fundamental logical principles.

In pure mathematics, actual objects in the world of existence will never be in question, but only hypothetical objects having those general properties upon which depends whatever deduction is being considered, and these general properties will always be expressible in terms of the fundamental concepts which called logical constants. A mathematical concept is logically analyzed and defined in the theoretical Platonic universe, because every definition is equal to a certain logical expression. Thus, a definition functions just as an abbreviation of a lengthy expression for typographical convenience and theoretically it is unnecessary ever (Whitehead & Russell, 1935).

Whitehead and Russell (1935) rejected classical Aristotelian logic and adopted a newly developed symbolic logic, so that deduction becomes a formal procedure operating on symbolic expressions. They considered symbolic logic as an aid to achieving strictly accurate demonstrative reasoning. Therefore, proof becomes a process of reasoning being represented as a valid derivation in symbolic logic. The first role of proof in the logician program is to show that certain propositions are tautological. Logicians aim to show that their axiomatic system is powerful enough to deduce these propositions.

It must be remembered that we are not affirming merely that such and such propositions are true, but also that the axioms stated by us are sufficient to prove them. In mathematics, the greatest degree of self-evidence is usually not to be found quit at the beginning, but at some later point. Hence the early deductions, until they reach this point, give reasons rather for believing the premises because true consequences follow from them, than for believing the consequences because they follow from the premises. The main concern of logicians is to deduce ready-proved theorems of ordinary mathematics from an axiomatic system. The function of the logicians' proof is to demonstrate the strength of their axiomatic system.

Formalism contends that the structural of proof concerns only the formalized axiomatic systems. Curry (1951) said that mathematics is the science of formal system. In this point of view, the mathematical proof of a proposition P is described as a sequence of propositions P_1, P_2, …, P_n (=P) such that P_i is an axiom, or a theorem, or is derived from previous propositions by rules of inference. This concept supported by Bourbaki (1968). He mentioned that as for as reading or writing a formalized text is concerned, it matters little whether this or that meaning is attached to the words, or signs in the text, or indeed whether any meaning at all is attached to them. The only important point is the correct observance of the rules of syntax.

The procedure involved in proof is conceived of simply as symbolic manipulation, where no consideration is accorded the meaning of the systems. The motivation for the formalization of proof is that its advocates want to establish absolute rigor in the proof procedure. They argue that if mathematics is related to Platonic entities or mental constructions, then intuition or ordinary reasoning may be involved, and these are often vague and subjective. Reducing the proof procedure to symbolic manipulation by removing any metaphysical concern from mathematics, an objective standard of proof can be achieved.

Mathematics had long been conceptualized as discipline independent of sociological explanation. It seemed to be impossible to detect any sociological trace within the purely logical appearance of this field. Though the sociology of mathematicians can be easily understood, that of mathematical knowledge seemed to be immune from investigation. Bloor (1976) noted that even originators of the sociology of knowledge conceived mathematics to be so. Recent work in the philosophy of mathematics, however, has come to reveal sociological aspects of mathematical knowledge. The major breakthrough came from an epistemological revolution.

Traditionally, mathematical knowledge was conceived of as a priori or analytic truth, which is absolute truth. Though Nagel(1950) denied the traditional view and contended that mathematical knowledge is inductive generalization, he supposed the certainty of induction, so that mathematical knowledge is always truth. As long as mathematical knowledge is conceptualized as absolute truths, mathematical practice consists simply in the accumulation of eternal truths. Consequently, the only task of sociology is the investigation of how certain mathematical truths were discovered. This effort, however, does not touch the nature of mathematical knowledge itself.

The totality of so-called knowledge or beliefs, from the most casual matters of geography and history to the profoundest laws of atomic physics or even of pure mathematics and logic, is a man-made fabric that impinges on experience only along the edges. A conflict with experience at the periphery occasions readjustments in the interior of the field. The values of truth have to be redistributed over some of our statements (Quine, 1961). This statement is a traditional distinction between synthetic and analytic knowledge and a holistic and relativistic view of knowledge.

After the breakdown of the foundations programs, several philosophers of mathematics have proposed new views of mathematical knowledge. In 1961 Lakatos asserted that mathematical knowledge is fallible, and by using historical case studies he conceptualized mathematics as a socially interactive construction. From the 1950s, mathematics considered as social practice in language game theory. In 1976 Bloor proposed a framework for the sociology of knowledge and applied it to logical and mathematical knowledge.

This sociological trend has been given impetus from a parallel trend in the philosophy of science. Thomas Kuhn(1962) argued against the view that science progresses by the simple accumulation of scientific truths. He proposed the paradigm shift in the history of science and pointed out the sociological nature of scientific knowledge. This theory has influenced the philosophy of both science and mathematics.

On the other hand, intuitionists opposed a structural view of proof, and advocating a psychological view of mathematical proof. They spurned the Platonic ontology. They contended that mathematical object exist in the human mind and are constructed by intuition. This intuition meant the basal intuition or intuition of two-oneness – the intuition of a whole as potentially to be divided into two parts (Brouwer, 1983). In the intuitionist's view a mathematical proposition is demonstrated as true only when a mental construction by one's basal intuition show the proposition to be true. Logic and language are considered to occupy a secondary status in proof. They are used only to describe mathematical thought for remembering or communicating, and that description is considered to be always incomplete. Therefore, proof is the very same mental construction.

The ontological status of the mathematical object has long been argued among philosophers of mathematics. Formalism fails to reflect the working mathematicians' perceptions such practitioners believe that they deal with something behind the symbols they use. But the something is not a physical object. It can be considered as a psychological phenomenon. Mathematicians do not agree with intuitionism. They do not consider a mathematical object to be just a mental construction in an individual mind because that seems too unstable to be a component of scientific theory.

In the sociology of mathematics, a mathematical object is considered to be a social construction. It is located at a concept within a theory. The theory concerns not just personal beliefs, or a part of culture (Bloor, 1976). A mathematical object, therefore, has a relatively stable and objective appearance.

In the psychological view, the axioms of a mathematical theory are supposed to consist of a set of propositions such that:

1. Each proposition of the set is true, and
2. The set of propositions is necessary and sufficient to deal with all the important issues of the theory.

However, in the structural view, (1) is replaced by the propositions of the set are consistent with each other. Both views of conceptions of an axiomatic system ignore the social aspects of mathematics. In this view, whether a proposition is trivially true depends on the dominant theory of a mathematical community. For example, in the 19th century the basis of trivial truths was changed from geometrical intuition to arithmetic intuition and then to intuition of sets. Likewise, the intuitionists' claim that only the intuition of two-oneness is legitimate may be acceptable in only temporal and limited ways. The present axiomatic systems have emerged through social dialectical processes.

In the concept of proof, definition is very important. Definition is a process in which the use of a word is explained by somebody and for somebody for various purposes. The role of definition is removal of ambiguity, change of meaning, isolation of new concepts, and so on. That is, definition is a language - based field. This conception of definition also fits mathematical definitions. Informal mathematics demonstrated that definitions are social acts (Sekiguchi, 1991).

Informal mathematics uses those concepts whose boundaries or extensions are vague. When those boundary objects or anomalies are found, where to draw a borderline of the concept can be a problem. When those anomalies that appear to be instances of a concept seem to contradict or refute a conjecture, belief, or theory, the problem becomes serious.

Defining is one of the strategies used to cope with refutations. The strategies in which a term being used in a conjecture, belief, or theory is defined in an ad hoc way so as to expel counterexamples. It is open used in mathematics as well as in everyday life. The definition is not a description of an essence acquired beforehand but a post hoc invention about a class having family resemblances (Bloor, 1983). The definition accompanies isolating and naming new concepts and introducing them to other people. A definition may not determine the extension of a concept once and for all. When a novel object appears, social negotiation is held as to whether it satisfies the definition, based on an understanding of the object, the past application of the definition, and the purpose of the inquiry (Sekiguchi, 1991).

CONCLUSION

In the sociological perspective, proof has to be understood from the human practices involved. The practice of proof seems to involve three different but closely interconnected processes: a quest for a proof, an organization of the proof, and an explanation of the proof to other people. In the request for a proof, a mathematician analyzes the present problem, conjecture, and some previous proofs and examines whether the conjecture is true and how it could be deduced from known theorems. This process ends by finding a proof or refuting the conjecture. In the former case, the analysis is followed by the organization of the proof: the process of arranging the results of analysis into a deductively articulated argument. The deductively organized proof is explained to other people through lectures or publications. However, a proof becomes a proof only when it is accepted in a mathematical community (Manin, 1977).

Therefore, explaining a proof to other people is not just presenting a proof. The mathematician has to try to convince other people. Here attention is required as to the social context of explaining. The original proposition to be proved may be modified through the interaction with other members of the community. Thus, the discovery of a proposition does not necessarily precede its proof.

The preceding three processes overlap and affect each other. Sometimes, any one of these processes is called a proof. Depending on the emphasis, the labels of proving are different. The nature of proof is described as a test or thought experiment when the analysis is meant a verification or justification when the synthesis is meant, and convincing people or a message when the explanation to other people is meant.

The preceding labels concern only what each process concludes whether a conjecture is true or false and whether the proof convinces people. However, the most important feature of proving is overlooked. Each proof is constructed in this way but not in other way (Wittgenstein, 1978). That is each act of proving accompanies an invention of a particular theoretical framework in which the conjecture in question is to be embedded (Lakatos, 1976).

According to Lakatos's case study, Cauchy's proof of Euler's conjecture on polyhedra gave birth to the rubber sheet theory, which formed a view of polyhedra completely different from the theory of solids. Unlike the latter, the former does not concern angles, lengths, ratios, shapes of lines and faces, and so forth but connections among vertices.

Wittgenstein (1978) points out that the proof changes the grammar of our language, changes our concepts. It makes new connections, and it creates the concept of these connections. As a result, a conjecture is understood differently than before. In terms of Lakatos (1976), the conjecture has been proved in several different ways, it receives several different interpretations. Thus, different proofs yield different theorems.

Once the conjecture having been proved faces refutations or criticism, those assumptions constituting the interpretations are disclosed and articulated (von Glasersfeld, 1983). Because those differences in understanding of a conjecture are not formulated explicitly, they are usually unnoticed. In Lakatos's case study on Euler's conjecture mentioned previously, for example, such assumptions as simple and simply-connected, inherent in the rubber sheet theory of Cauchy's proof, where isolated and developed into basic constructions of theory after counterexamples were proposed.

Psychological, structural, and social perspective of proof has different positions. Criticism of a proof was not conceived in the psychological perspective of proof as well as structural perspective of proof. Because the model of mathematical practice in traditional mathematical philosophy is one ideal mathematician, absolute criteria and infallible intuition are supposed for judging the validity of proof. However, from the sociological perspective, the criteria and intuition for judge of proof are community dependent. Who prove a theorem have to correct their proofs through feedback from their communities.

REFERENCES

Bauersfeld, H. (1988). Interaction, construction, and knowledge: Alternative perspectives for mathematics education, In D. A. Grouws, T. J. Cooney, & D. Jones (Eds.), *Perspectives on research on effective mathematics teaching: Vol. 1* (pp. 27-46). Reston, VA:National Council of Teachers of Mathematics.

Bittinger, M. L. (1968). The effect of a unit in mathematical proof on the performance of college mathematics majors in future mathematics courses. *Dissertation Abstracts*, *29,* 3906A.

Bloor, D. (1976). *Knowledge and social imagery*. London: Routledge & Kegan Paul.

Bloor, D. (1983). *Wittgenstein: A social theory of knowledge.* New York: Columbia University Press.

Bourbaki, N. (1968*). Elements of mathematics: Theory of sets*. Paris: Hermann.

Brennan, C. M. (1976). Toward a theory of sequencing: Study 4-3: the development and investigation of a canonical teaching procedure for aspects of mathematical proof. *Dissertation Abstracts International, 37,* 7093A.

Brouwer, L. E. J. (1983). Intuitionism and formalism. In P. Banacerraf & H. Putnam (Eds.), *Philosopy of mathematics* (2nd ed.) (pp. 77-89). Cambridge: Cambridge University Press.

Carlson, P. R. (1971). An investigation of the effects of instruction in logic on pupils' success in proving theorems in mathematics. *Dissertation Abstracts International, 35,* 7765A.

Confrey, J. (1987). The constructivist. In J. C. Bergeron, N. Herscovics, & c. Kieran (Eds.), *Proceedings of the Eleventh International Conference of Psychology of Mathematics Education: Vol. 3* (pp. 307-317). Montreal: University de Montreal.

Curry, H. B. (1951). *Outlines of a formalist philosophy of mathematics*. Amsterdam, Netherlands: North-Holland.

Cussins, A. (1987). Varieties of psychologism. *Synthese, 70*, 123-154.

Frege, G. (1950). *The foundations of arithmetic*. Oxford: Basil Blackwell.

Khun, T. (1962). *Paradigm shift*.

Lakatos, I. (1976). *Proofs and refutations*. Cambridge, England: Cambridge University Press. (Based on PhD dissertation of 1961)

Lakatos, i. (1978). Infinite regress and foundations of mathematics. In J. Worrall & G. Currie (Eds.), *Imre Lakatos, Philosophical papers: Vol. 2. Mathematics, science and epistemology* (pp. 3-23). Cambridge, England: Cambridge University Press.

Manin, Y. I. (1977). *A course in mathematical logic*. New York: Springer-Verlag.

Nagel, E. (1950). *John Stuart Mill's philosophy of scientific method*. New York: Hafner.

Popper, K. R. (1950). *The open society and its enemies*. Princeton, NJ: Princeton University Press.

Quine, W. V. O. (1961). On what there is. In W. V. O. Quine, *From a logical point of view* (2nd ed., pp. 1-19). Cambridge, Ma: Harvard University Press.

Russell, B. (1937). *The principles of mathematics*(2nd ed.). London: George Allen & Unwin.

Sekiguchi, H. (1991). *An investigation on proofs and refutations in the mathematics classroom*. Unpublished doctoral dissertation, University of Georgia, Athens

Senk, S. L. (1985). How well do students write geometry proofs? *Mathematics teacher, 78*, 448-456

Steffe, L. P. & Cobb, P. (1983). The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education, 14*, 83-94.

Summa, D. J. (1981). The effect of proof format, problem structure, and the type of given information on achievement and efficiency in geometric proof. *Dissertation Abstracts International, 42*, 3084A.

Tymoczko, T. (1985). *New directions in the philosophy of mathematics*. Boston: Birkhuser.

Von Glasersfeld, E. (1987). Learning as constructive activity. In E. von Glasersfeld, *The construction of knowledge: Contributions to conceptual semantics* (pp. 307-333). Seaside, CA: Intersystems Publications.

Weinberg, D., & Gavelek, J. (1987). A social constructivist theory of instruction and the development of mathematical cognition. In J. C. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the Eleventh International Conference of Psychology of Mathematics Education: Vol. 3* (pp. 307-317). Montreal: University de Montreal.

Whitehead, A. N., & Russell, B. (1935*). Principia mathematica: Vol. 1* (2nd ed.). London: Cambridge University Press.

Williams, E. R. *1979). *An investigation of senior high school students' nuderstanding of the nature of mathematical proof*. Unpublished doctoral dissertation, University of Alberta, Edmonton.

Wittgenstein, L. (1978). *Remarks on the foundations of mathematics* (2nd ed.). Cambridge, Ma: MIT Press.