GENERALIZATION AND CONSTRUCTION: BEAUTIES AND PHILOSOPHY IN MATHEMATICS

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In memory of Dr. W. X. Shi

ABSTRACT. This article gives an overview of mathematics and presents some typical examples of its development. We mainly show some beauties of mathematics and discuss some philosophies in mathematical research.

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1. INTRODUCTION

Much of mathematics often has a great aesthetic aspect to those who understand, create, discover, and appreciate mathematics. Many mathematicians speak of the inherent beauty of mathematics, its true aesthetics, and its hidden beauty. Simplicity and generality are highly

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valued, and there is beauty in simple and elegant proofs, such as Euclid's short proof that there are infinitely many primes, in beautiful and useful theorems, such as Brouwer's fixed point theorem and the maximum principle for partial differential equations, and in algorithms that are fundamental to numerical computation, such as Euclidean algorithm, Newton's method, and gradient descent. There are articles on good mathematics (see, for example, [23]) and on cultures of mathematics (see, for example, [9]).

This article gives an overview of mathematics and presents some typical examples of its development. We mainly show some beauties of mathematics and discuss some philosophies in mathematical research.

2. Generalization: The Trend in Mathematics

Let us start with a concrete example to see how the principle of generalization has developed in mathematics. As we know, the Fundamental Theorem of Calculus establishes the relationship between a differentiable function and its derivative. The first step in the process of generalizations is about the dimension. Generalization to 2-dimensional regions in Euclidean space, as we know, leads to Green's theorem, which is widely and frequently used in calculus and in partial differential equations. In $\mathbb{R}^3$, it corresponds to the Gauss-Ostrogradsky theorem. The definition of a manifold is a generalization of Euclidean space. As is well known in differential topology, Stokes' theorem can be simply formulated in this general setting as

$$\int_{M^n} d\omega = \int_{\partial M^n} \omega, \tag{2.1}$$

where ω is an $(n-1)$-form for n-dimensional manifold M^n. If we look back to the Fundamental Theorem of Calculus at this point, it is indeed a special case of Stokes' theorem, since the boundary of an interval in $\mathbb{R}^1$ consists of two points. Since both sides of (2.1) are functionals on differential forms, it can be expressed in the notation of functional analysis (see, for example, [7]) as follows

$$\langle d\omega, M \rangle = \langle \omega, \partial M \rangle . \tag{2.2}$$

Furthermore, and more generally, as the theory of de Rham's has developed, the general boundary operator has become a unified operator that can be applied not only to functions but also to manifolds or complexes, namely the boundary operator for complexes. Thus, another way to formulate the generalized Stokes theorem in an extremely simple way is the following:

Theorem 2.1. *The boundary operator as an elliptic operator is self-adjoint.*

The beauties of mathematics are illustrated in a way that is simple in form, but contains rich connotations in various theorems and formulas, such as the Gauss-Bonnet Theorem, an important theorem connecting the topological characteristic and the geometric property for manifolds without boundary as

$$\frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}}\int_{M^n} KdV = \chi(M^n), \tag{2.3}$$

where K is the Lipschitze-Killing curvature on the n-dimensional manifold M^n, and for manifolds with boundary as

$$\frac{\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}}}\int_{\partial M^n} k_1k_2\cdots k_{n-1}dx = \chi(M^n), \tag{2.4}$$

where $k_1, k_2, \cdots, k_{n-1}$ are principal curvatures at x of ∂M^n. These formulae have very important generalizations, such as intrinsic volumes, in integral geometry (see, for example, [17]), which form a basis for the valuation space on space forms (see, for example, [5], [13], [14], [16], and [15]).

It seems that the pursuit of simplicity has driven and continues to drive the development of mathematics. For example, the Atiyah-Singer Index Theorem states that the analytic index of an elliptic operator and its topological index are identical. In particular, the Atiyah-Singer Index Theorem is a generalization of the Gauss-Bonnet theorem in a broad and far-reaching sense, as well as a generalization of the Riemann-Roch theorem, to some extent (see, for example, [2], [3], [1], and [4]).

Similarly, the principle of simplicity is also applied in physics, as in Einstein's famous Mass-Energy equation

$$E = mc^2 \tag{2.5}$$

which is a very simple expression but has profound meaning, rich content, and great significance.

Most developments in mathematics were brought about or driven by the idea of generalizations. We can summarize the common generalizations that drive the development of mathematics as follows:

(1) From low dimension to high dimension, the examples of this point are innumerable;
(2) From the finite to the infinite, for example, the basic linear algebra to functional analysis, in which the objects we deal with range from finite-dimensional to infinite-dimensional;
(3) From the fixed to the varying, for example, the Crofton-Poincare formula to the principal kinematic formulae, one of the most remarkable developments in the history of integral geometry;
(4) From the discrete to the continuous, for example, the discrete sum to the theory of integral calculus;
(5) From the concrete to the abstract, for example, linear algebra to the theory of groups, rings, modules, and fields in abstract algebra;
(6) From more conditions to fewer conditions, one of the most notable examples being the definition of T_4, $T_{3\frac{1}{2}}$, T_3, $T_{2\frac{1}{2}}$, T_2, T_1, and T_0 spaces in topology;
(7) From objects having nice properties everywhere to objects having not nice properties somewhere, which is exemplified in various singularity theories in many areas of mathematics;
(8) and so on.

The point of emphasis on noticing generalizations is that many complicated and profound concepts or theories look more natural or simple when we look at them in this way. So we can try to understand simple cases first in order to grasp a deep theorem, in mathematical teaching introduce the classical version of a theorem or concept first before explaining the generalized version, and in mathematical research start from working on particular examples.

3. Constructions in Mathematics

Another beautiful aspect of mathematics is constructions, which are in some ways different from generalization. Constructions are very creative work, and many constructions have led to breakthroughs in the development of mathematics. For example, the solution of Poincare Conjecture is established and based on Hamilton's construction of the nonlinear Ricci flow equation (see for example [18] and [6]),

$$\frac{\partial g(t)}{\partial t} = -2\mathrm{Ric}\left(g\left(t\right)\right), \tag{3.1}$$

where $g\left(t\right)$ is the Riemannian metric varying with time t on a manifold and *Ric* is the Ricci curvature on the manifold. This equation describes the relationship between the metrics on a manifold and their curvatures. Hamilton introduced this equation, originally called evolution equation (see [11]), in order to study 3-dimensional manifolds and the Poincare Conjecture. Following the program initiated by Hamilton, based on the foundational work on complete non-compact and its related work (see for example [22], [8], and [12]), Perelman solved the problem arising in the singularities case, and this implies the complete solution of the Poincare Conjecture and, moreover, of the general Thurston's Geometrization Conjecture (see, for example, [19], [21], and [20]). Looking back over the history of the whole development, we can see the fundamental and great importance of the construction of Hamilton's Ricci flow equation.

According to Gowers' criteria on mathematical culture (see, for example, [10]), solving the Poincare Conjecture is, of course, a process of problem solving, as was the motivation of Hamiltion's Ricci Flow equation. But it is also a very large project for building and understanding theories, the Ricci flow, and the structure of 3-dimensional manifolds.

4. A Summary

From a philosophical point of view, it is striking that the principle of generalization dominates the whole of mathematics, while the trend towards unification runs through the entire development of mathematics. Looking at mathematics as a whole, generalization and construction have occurred alternately in the development of mathematics. Construction is like various revolutions in human history, while generalization is like the propagation of theories or ideologies. Nevertheless, both of them have given mathematics a great boost or push of development and progress and have led mathematics to great heights of development and progress.

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Declaration of interest

The author declares that there is no conflict of interest.

Data Availability Statement

The author confirms that the data supporting the findings of this study are available within the article or its supplementary materials.

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